MATH 2020 B

April 13, 2022

Surface integral of functions

$\vec{r} : D \to \mathbb{R}^3$ surface S

f a function on S

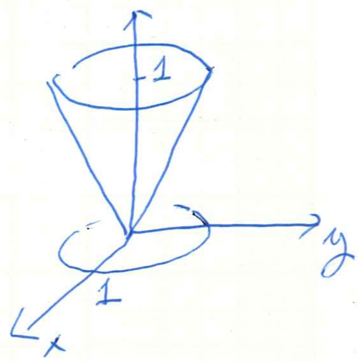
(explain later)

$$\iint_S f \, d\sigma \overset{\text{def}}{=} \iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA(u,v)$$

Meaning : ① $f \geq 0$, it gives the mass of S with density f

② $f \equiv 1$, it give the surface area.

e.g. Evaluate $\iint_S x^2 \, d\sigma$, S $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$



S is a graph, so

$$\underset{D_1}{(x,y)} \longmapsto (x, y, \sqrt{x^2+y^2})$$

$\vec{r}(x,y) = x\hat{i} + y\hat{j} + \sqrt{x^2+y^2}\,\hat{k}$

$\vec{r}_x \times \vec{r}_y = \dfrac{-x}{\sqrt{x^2+y^2}}\hat{i} + \dfrac{-y}{\sqrt{x^2+y^2}}\hat{j} + \hat{k}$ standard formula for a graph

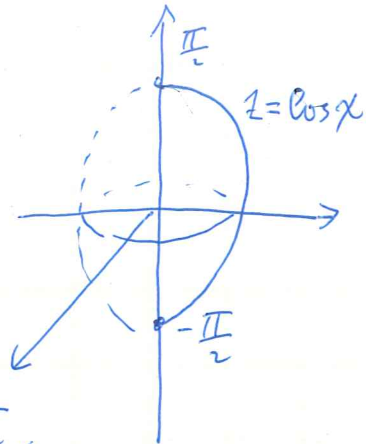
$|\vec{r}_x \times \vec{r}_y| = \sqrt{2}$

$$\therefore \iint_S x^2 \, d\sigma = \iint_{D_1} x^2 \sqrt{2} \, dA(x,y)$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 r^2 \cos^2\theta \, r \, dr \, d\theta$$ (change to polar)

$$= \pi\sqrt{2}/4 \ \#$$

e.g. Find $\iint_S \sqrt{1-x^2-y^2}\, d\sigma$, S: rotate $x = \cos z$ around z-



standard formula

$$\vec{r}\,(z,\alpha) = \cos z \cos\alpha\, \hat{i} + \cos z \sin\alpha\, \hat{j} + z\, \hat{k}$$

$$\vec{r}_z \times \vec{r}_\alpha = \cos z \sqrt{1+\sin^2\alpha}$$

$$\iint_S \sqrt{1-x^2-y^2}\, d\sigma = \iint_{[0,2\pi]\times[-\pi/2,\pi/2]} \sqrt{1-(\cos z\cos\alpha)^2-(\cos z \sin\alpha)^2} \times |\cos z \sqrt{1+\sin^2\alpha}|$$

$0 \le \alpha \le 2\pi$

$-\frac{\pi}{2} \le z \le \pi/2$

$$= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} |\sin z| \cos z \sqrt{1+\sin^2 \cancel{z}}\, dz$$

$t = 1+\sin^2 z$

$dt = 2\sin z \cos z$

$$= 4\pi \int_1^2 \frac{1}{2}\sqrt{t}$$

$$= \frac{4\pi}{3}(2\sqrt{2}-1) \ \#$$

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surface integral of vector fields.

$$\iint_S \vec{F}\cdot d\vec{\sigma} \overset{\text{def}}{=} \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA(u,v)$$

$$\overset{\text{or}}{=} \iint_D \vec{F}(\vec{r}(u,v)) \cdot \hat{n}\, d\sigma$$

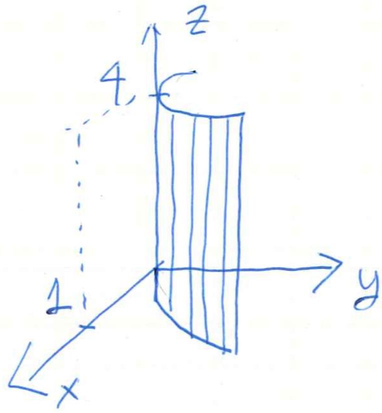
There are 2 choices of the normal vector, choose one to determine the orientation of S.

$$\hat{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

$$\left(\begin{aligned} \vec{r}_u \times \vec{r}_v \cdot \vec{r}_u = 0 \\ \vec{r}_u \times \vec{r}_v \cdot \vec{r}_v = 0 \end{aligned} \right)$$

meaning : the flux of $\vec{F}$ across S (in its normal direction).

e.g. Find the flux of $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, where $\hat{n}$ points in $-y$-direction



$$(x, z) \longmapsto (x, x^2, z)$$

$[0,1] \times [0,4]$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2x\hat{i} - \hat{j} + 0\hat{k}.$$

At $(0,0,0)$, $= (0,-1,0)$ points in $-y$-directi..

$$\therefore \hat{n} = \frac{2x\hat{i} - \hat{j}}{\sqrt{4x^2+1}}$$

flux $\displaystyle\iint_S \vec{F} \cdot d\vec{\sigma} = \iint_{[0,1]\times[0,4]} (x^2 z\hat{i} + x\hat{j} - z^2\hat{k}) \cdot (2x\hat{i} - \hat{j} + 0\hat{k})\, dA(x,z)$

$$= \iint_{[0,1]\times[0,4]} (2x^3 z - x)\, dA(x,z)$$

$$= \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz$$

$\vdots$

$= 2$ #

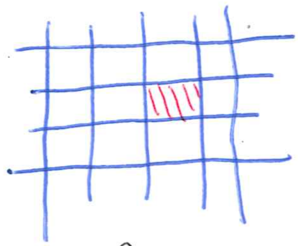
Now we explain why the mass of a surface S with density f is given by

$$\iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA(u,v).$$

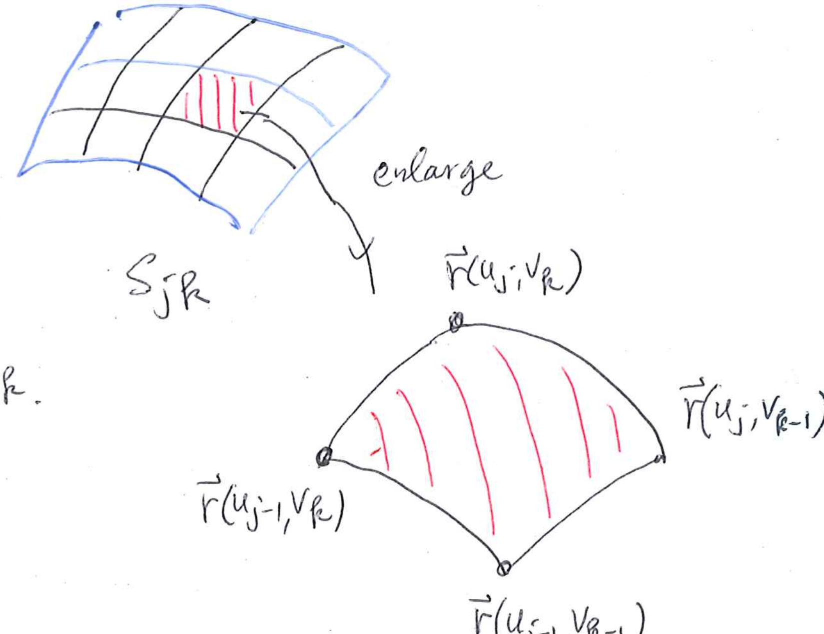
Consider $[a,b]\times[c,d] \xrightarrow{\vec{r}} S$ a parametrization for S.

P : $a \le u_0 < u_1 < \cdots < u_n = b$
$c \le v_0 < v_1 < \cdots < v_m = d$
R_{jk} subrectangles

S_{jk} = image of R_{jk} under $\vec{r}$



R_{jk}



S_{jk}

approximate mass :

area of S_{jk}.

$$\sum_{j,k} f(\vec{r}(u_{j-1}, v_{k-1}))\,|S_{jk}|$$

Use

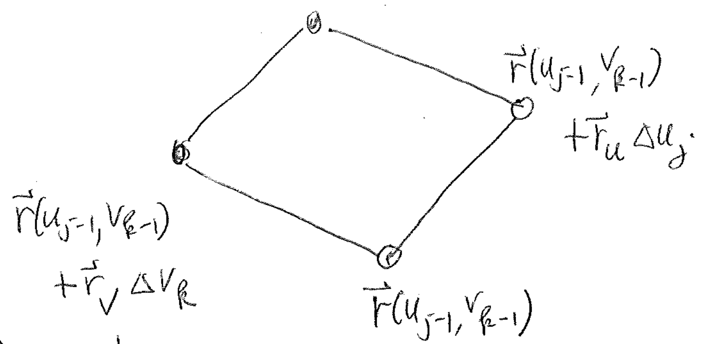
$$\begin{aligned}\vec{r}(u_j, v_{k-1}) &= \vec{r}(u_{j-1}+\Delta u_j, v_{k-1})\\ &= \vec{r}(u_{j-1}, v_{k-1}) + \vec{r}_u(u_{j-1}, v_{k-1})\Delta u_j + \tfrac{1}{2}\vec{r}_{uu}(\Delta u_j)^2\\ &\approx \vec{r}(u_{j-1}, v_{k-1}) + \vec{r}_u(u_{j-1}, v_{k-1})\Delta u_j,\end{aligned}$$

Similarly,

$$\begin{aligned}\vec{r}(u_{j-1}, v_k) &= \vec{r}(u_{j-1}, v_{k-1}+\Delta v_k)\\ &\approx \vec{r}(u_{j-1}, v_{k-1}) + \vec{r}_v(u_{j-1}, v_{k-1})\Delta v_k,\\ \vec{r}(u_j, v_k) &= \vec{r}(u_{j-1}+\Delta u_j, v_{k-1}+\Delta v_k)\\ &\approx \vec{r}(u_{j-1}, v_{k-1}) + \vec{r}_u(u_{j-1}, v_{k-1})\Delta u_j + \vec{r}(u_{j-1}, v_{k-1})\Delta v_k,\end{aligned}$$

S_{jk} can be approximated by the parallelogram:

So $|S_{jk}|$ can be approximated by the area of this parallelogram, which is equal to



$$\left| \vec{r}_u(u_{j-1}, v_{k-1}) \Delta u_j \times \vec{r}_v(u_{j-1}, v_{k-1}) \Delta v_k \right|$$

$$= \left| \vec{r}_u(u_{j-1}, v_{k-1}) \times \vec{r}_v(u_{j-1}, v_{k-1}) \right| \Delta u_j \Delta v_k .$$

$$\therefore \quad \sum_{j,k} f(\vec{r}(u_{j-1}, v_{k-1})) \, |S_{jk}| \approx \sum f(\vec{r}(u_{j-1}, v_{k-1})) \, |\vec{r}_u \times \vec{r}_v| (u_{j-1}, v_{k-1}) \Delta u_j \Delta v_k$$

$$\to \iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| (u,v) \, dA(u,v) .$$

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A surface is called <u>orientable</u> if it admits a continuous unit normal vector field. A surface with a choice of a continuous vector field is called an <u>oriented</u> <u>surface</u>.

<u>Theorem</u> A regular parametric surface is orientable. It has 2 choices of orientation, given either by

$$\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \text{or} \quad -\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} .$$

Pf: Let $\vec{r}: D \to S$ be a regular parametrization of S.

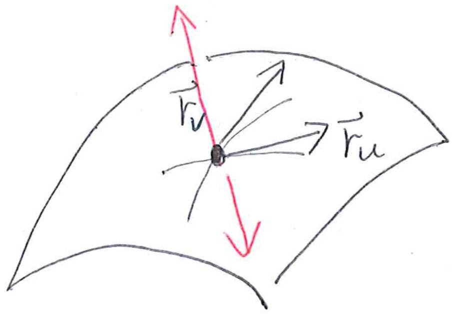
As $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ on D, $\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \neq \vec{0}$ has unit length.

Moreover, recall the property of cross product,

$$(\vec{a}\times\vec{b})\cdot\vec{a}=0, \quad (\vec{a}\times\vec{b})\cdot\vec{b}=0.$$

So, $\vec{r}_u\times\vec{r}_v\cdot\vec{r}_u=0$, $\vec{r}_u\times\vec{r}_v\cdot\vec{r}_v=0$, ie

$\vec{r}_u\times\vec{r}_v$ is perpendicular to $\vec{r}_u, \vec{r}_v$, & so it points to the normal direction.



$\pm \dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$ is a unit normal vector.

Clearly, there are only 2 choices of continuous normal unit v.f. on S: either

$$\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|} \quad \text{or} \quad -\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}.$$

Note: there are non-orientable surfaces.
In particular, these surfaces can't be regular parametric surfaces.
See Wiki, Möbius band for an example.